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Dynamical Wave Function Collapse Models in Quantum Measure Theory

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Abstract

The structure of Collapse Models is investigated in the framework of Quantum Measure Theory, a histories-based approach to quantum mechanics. The underlying structure of coupled classical and quantum systems is elucidated in this approach which puts both systems on a spacetime footing. The nature of the coupling is exposed: the classical histories have no dynamics of their own but are simply tied, more or less closely, to the quantum histories.

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1 Introduction

Models of "spontaneous localisation" or "dynamical wavefunction collapse" are observer independent alternatives to standard Copenhagen quantum theory (see [1] for a review). These models have a generic structure: there is a quantum state Ψ which undergoes a stochastic evolution in Hilbert space and there is a "classical" (c-number) entity – call it α – with a stochastic evolution in spacetime. The stochastic dynamics for the two entities – Ψ and α – are coupled together. The stochastic dynamics in Hilbert space tends to drive Ψ into an eigenstate of an operator $\hat{\alpha}$ that corresponds to α. And the probability distribution for the realised values of α depends on Ψ so that the history of α follows, noisily, the expectation value of $\hat{\alpha}$ in Ψ.

That collapse models have both quantum and classical aspects has been pointed out before, notably by Diósi. The nature of this interaction between the classical and quantum parts of these models is, however, somewhat obscured by the profound difference in the nature of their descriptions: the classical variable traces out a history in spacetime and the quantum state traces out its evolution in Hilbert space.

In order to illuminate the nature of the quantum-classical coupling within collapse models we will, in the case of a concrete and specific example, recast the formalism into the framework of *generalised measure theory* [2] in which both classical and quantum systems are treated on as equal a footing as possible. The classical variables will continue to have a spacetime description but the quantum system will now also be described in terms of its spacetime histories and not fundamentally in terms of any state in Hilbert space.

The model we will focus on is a discrete, finite, 1+1 dimensional lattice field theory. This is a useful model because it is completely finite (so long as we restrict ourselves to questions involving finite times) and expressions can be written down exactly and also because there is a well-defined background with non-trivial causal structure, so that questions of causality can be explored.

We will show that the model contains both "classical" and "quantum" histories, and demonstrate the nature of their interaction. We will show that one choice of ontology for collapse models, the Bell ontology [3], corresponds to coarse graining over the quantum histories. We will also show how the well-known relationship between collapse models and open quantum systems coupled to an environment reveals itself in this histories framework.

2 Quantum measure theory

We start with a brief review of generalised measure theory and quantum measure theory and refer to [2, 4, 5, 6, 7] for more details.

A generalized measure theory consists of a triple, $(\Omega, \mathfrak{A}, \mu)$, of a space of histories, an event algebra and a measure. The space of histories, Ω, contains all the "fine grained histories" or "formal trajectories" for the system *e.g.* for n-particle mechanics – classical or quantum – a history would be a set of n trajectories in spacetime, and for a scalar field theory, a history would be a field configuration on spacetime.

The event algebra, $\mathfrak{A}$, contains all the (unasserted) propositions that can be made about the system. We will call elements of $\mathfrak{A}$ *events*, following standard terminology in the theory of stochastic processes. In cases where Ω is finite, $\mathfrak{A}$ can be identified with the power set, 2^Ω. When Ω is infinite, $\mathfrak{A}$ can be identified with an appropriate ring of sets contained in the power set: $\mathfrak{A} \subset 2^\Omega$.[1]

Predictions about the system — the dynamical content of the theory — are to be gleaned, in some way or another, from a generalized measure μ, a non-negative real function on $\mathfrak{A}$. μ is the dynamical law and initial condition rolled into one.

Given the measure, we can construct the following series of symmetric set functions,which are sometimes referred to as the Sorkin hierarchy[2]:

$$
\begin{aligned}
I_1(X) &\equiv \mu(X) \\
I_2(X,Y) &\equiv \mu(X \sqcup Y) - \mu(X) - \mu(Y) \\
I_3(X,Y,Z) &\equiv \mu(X \sqcup Y \sqcup Z) - \mu(X \sqcup Y) - \mu(Y \sqcup Z) - \mu(Z \sqcup X) \\
&\quad + \mu(X) + \mu(Y) + \mu(Z)
\end{aligned}
$$

and so on, where X, Y, Z, *etc.* are disjoint elements of $\mathfrak{A}$, as indicated by the symbol '$\sqcup$' for disjoint union.

A *measure theory of level k* is defined as one which satisfies the sum rule $I_{k+1} = 0$. It is known that this condition implies that all higher sum rules are automatically satisfied, *viz.* $I_{k+n} = 0$ for all $n \geq 1$ [2].

A level 1 theory is thus one in which the measure satisfies the usual Kolmogorov sum rules of classical probability theory, classical Brownian motion being a good example. A level 2 theory is one in which the Kolmogorov sum rules may be violated but I_3 is nevertheless zero. Any unitary quantum theory can be cast into the form

[1] $\mathfrak{A}$ is a Boolean algebra with addition in the algebra corresponding to symmetric difference and multiplication in the algebra corresponding to intersection. We will not employ this algebraic notation in this paper.

[2] These are the generalised interference terms introduced in [2]

of a generalised measure theory and its measure satisfies the condition $I_3 = 0$. We refer to level 2 theories, therefore, as *quantum* measure theories.

The existence of a quantum measure, μ, is more or less equivalent [2] to the existence of a *decoherence functional*, $D(\,\cdot\,;\,\cdot\,)$, a complex function on $\mathfrak{A}\times\mathfrak{A}$ satisfying [8, 9]:

(i) Hermiticity: $D(X\,;Y) = D(Y\,;X)^*$, $\forall X, Y \in \mathfrak{A}$;

(ii) Additivity: $D(X \sqcup Y\,;Z) = D(X\,;Z) + D(Y\,;Z)$, $\forall X, Y, Z \in \mathfrak{A}$ with X and Y disjoint;

(iii) Positivity: $D(X\,;X) \geq 0$, $\forall X \in \mathfrak{A}$;

(iv) Normalization: $D(\Omega\,;\Omega) = 1$.[3]

The quantal measure is related to the decoherence functional by

$$\mu(X) = D(X\,;X) \quad \forall X \in \mathfrak{A}\,. \tag{2.1}$$

The quantity $D(X\,;Y)$ is interpretable as the quantum interference between two sets of histories in the case when X and Y are disjoint.

3 The lattice field model

We review the lattice field model [10, 11] whose structure we will investigate. The model is based on a unitary QFT on a 1+1 null lattice [12], which becomes a collapse model on the introduction of local "hits" driving the state into field eigenstates.

The spacetime lattice is a lightcone discretisation of a cylinder, N vertices wide and periodic in space. It extends to the infinite future, and the links between the lattice vertices are left or right going null rays. Figure 1 shows a part of such a spacetime lattice, identifying the leftmost vertices with the rightmost vertices we see that $N = 6$. A spacelike surface σ is maximal set of mutually spacelike links, and consists of N leftgoing links and N rightgoing links cut by the surface; an example of a spatial surface is shown in figure 1. We assume an initial spacelike surface σ_0.

An assignment of labels, $v_1, v_2, v_3, \ldots$, to the vertices to the future of σ_0 is called "natural" if $i < j$ whenever the vertex labelled v_i is to the causal past of the vertex labelled v_j. A natural labelling is equivalent to a linear extension of the (partial) causal order of the vertices. A natural labelling, $v_1, v_2, \ldots$ is also equivalent to a sequence of spatial surfaces, $\sigma_1, \sigma_2, \ldots$ where the surface σ_n is defined such that between it and σ_0, lie exactly the vertices $v_1, \ldots v_n$. One can think of the natural labelling as giving an "evolution" rule for the spacelike surfaces: at time step n the

[3] The normalisation condition may turn out not to be necessary, but we include it because all the quantum measures we consider in this paper will satisfy it.

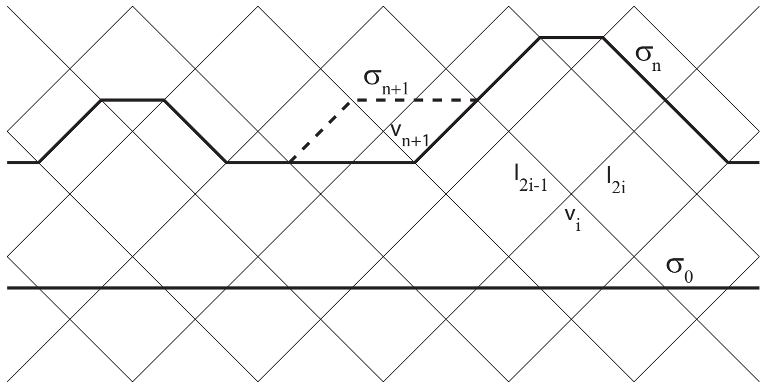

Figure 1: The light cone lattice. σ_0 is the initial surface and σ_n is a generic spacelike surface. The surface σ_{n+1} is shown after the vertex v_{n+1} is evolved over. A vertex v_i is shown with its two outgoing links: l_{2i-1} to the left and l_{2i} to the right.

surface creeps forward by one "elementary motion" across vertex v_n. For the purpose of this paper, it is convenient to consider a fixed natural labelling. Nothing will depend on the natural labelling chosen, all mathematical quantities will be independent of the choice.

The local field variables Φ live on the links. These field variables take only two values $\{0, 1\}$, so that on each link there is a qbit Hilbert space spanned by the two field eigenstates $\{|0\rangle, |1\rangle\}$. As the field variables live on the links, it is convenient to have a labelling of the links. We choose a labelling l_a, $a = 1, 2, \ldots$, such that l_{2i-1} and l_{2i} are the left-going and right-going outgoing links, respectively, from vertex v_i (see figure 1). So, as vertex label i increases from 1 to n, the link label a runs from 1 to $2n$. We denote the qbit Hilbert space related to link l_a by H_{l_a}.

The initial state $|\psi_0\rangle$ on surface σ_0 is an element of the 2^{2N} dimensional Hilbert space H_{σ_0} which is a tensor product of the $2N$ 2-dimensional Hilbert spaces on each link cut by σ_0, $H_{\sigma_0} = \bigotimes_{l_a \in \sigma_0} H_{l_a}$. Similarly there is a 2^{2N} dimensional Hilbert space for each spacelike surface σ_i and they are isomorphic via the isomorphisms, tied to the lattice, which map each link's qbit Hilbert space onto the Hilbert spaces for the links vertically above it on the lattice. In this way we can identify the Hilbert spaces H_{σ_i} $(= \bigotimes_{l_a \in \sigma_i} H_{l_a})$ on each surface and describe the time evolution with a state evolving in a single Hilbert space H_q $(\simeq H_{\sigma_i})$ of the system.

3.1 The unitary theory

In the standard unitary version of this local field theory, there is a local unitary evolution operator, R_i, for each v_i, which acts unitarily on the 4-dimensional factor

of the Hilbert space associated to the two ingoing and two outgoing links for v_i, and acts as the identity operator on all other factors. The state vector is evolved from σ_{i-1} to σ_i by applying R_i [12].

So in figure 1 we see that the surface σ_n evolves 'over' vertex v_{n+1} to give us surface σ_{n+1}. Now if l_j, l_k are the two links going 'into' vertex v_{n+1}, and $l_{2(n+1)-1}, l_{2(n+1)}$ the two outgoing links, the operator R_{n+1} maps $H_{l_j} \otimes H_{l_k}$ to $H_{l_{2(n+1)-1}} \otimes H_{l_{2(n+1)}}$. Further, for the links in the intersection of σ_n and σ_{n+1}, R_{n+1} acts as the the identity. Since the surfaces σ_n, σ_{n+1} only differ on the links $l_j, l_k, l_{2(n+1)-1}, l_{2(n+1)}$, we can put this together to get $R_{n+1} : H_{\sigma_n} \to H_{\sigma_{n+1}}$.

Since we have identified the Hilbert spaces H_{σ_i}, we regard R_{n+1} as evolving a state in the 'system Hilbert space' H_q, so we write:

$$\begin{aligned} |\psi_{n+1}\rangle &= R_{n+1}|\psi_n\rangle \\ &= R_{n+1}R_n \ldots R_1|\psi_0\rangle \,. \end{aligned} \tag{3.1}$$

We define the unitary time evolution operator, $U(n)$, by

$$U(n) \equiv R_n \; R_{n-1} \ldots R_1 \; . \tag{3.2}$$

To cast the theory into a quantum measure theory framework, we need to identify the space, Ω_q of histories, an event algebra, $\mathfrak{A}_q$, of suitable subsets of Ω_q and the decoherence functional, $D_q(\,\cdot\,;\,\cdot\,)$.

In the lattice field theory the set of histories, Ω_q, is the set of all field configurations on the lattice to the future of σ_0. A field configuration, Φ, is an assignment of 0 or 1 to every link, in other words Φ is a function from the infinite set of links, $\{l_a : a = 1, 2, \ldots\}$, to $\mathbb{Z}_2$.

The events that we want to consider are those which refer to properties of the histories which are bounded in time. In other words for $A \subset \Omega_q$ to be an event there must exist an integer m such that to determine whether or not a field configuration, Φ is in A it is only necessary to know the values of Φ between σ_0 and σ_m. For example, the subset

$$E_k = \{\Phi \in \Omega_q : \Phi(l_{2k}) = 1\}$$

is an event for any fixed k. But the subset

$$E = \{\Phi \in \Omega_q : \exists k \text{ s.t. } \Phi(l_{2k}) = 1\}$$

is *not* an event (at least not for the purposes of the current paper).

We want to consider all events that are bounded in time. To this end, for each positive integer n we define Ω_q^n to be the set of field configurations, Φ^n, on the

first $2n$ links, $l_1, \ldots l_{2n}$, outgoing from the first n vertices $v_1, \ldots v_n$. (Recall that we have chosen an arbitrary, but fixed, natural labelling of the vertices which gives unambiguous meaning to "the first $2n$ links".) We define the cylinder set $Cyl(\Phi^n)$ to be the set of all elements of Ω_q which coincide with Φ^n on $l_1, \ldots l_{2n}$:

$$Cyl(\Phi^n) \equiv \{\Phi \in \Omega_q | \Phi = \Phi^n \text{ when restricted to the first } 2n \text{ links}\} .$$

Each cylinder set, $Cyl(\Phi^n)$ is an event that is bounded in time: it is the event "the first $2n$ values of the field agree with Φ^n." The event algebra, $\mathfrak{A}_q$, then, is the (unital) ring of sets generated, under finite union and intersection, by all the cylinder sets, $Cyl(\Phi^n)$, for all n and all $\Phi^n \in \Omega_q^n$.

Two cylinder sets have nonempty intersection if and only if one contains the other and the complement of a cylinder set (that for Φ^n, say) is a disjoint union of finitely many cylinder sets (those for all the configurations on $l_1 \ldots l_{2n}$ that are *not* Φ^n). Thus, all elements of $\mathfrak{A}_q$ are finite, disjoint unions of cylinder sets. Given an event, $A \in \mathfrak{A}_q$, there is indeed an integer, m, such that to determine whether or not a field configuration, Φ is in A it is only necessary to know the values of Φ between σ_0 and σ_m. We will refer to the minimum such m as the *time extent* of A. The time extent of the cylinder set $Cyl(\Phi^n)$ is clearly n and the time extent of an event A is no greater than the maximum of the time extents of the cylinder sets whose union A is.

Consider the example given previously, E_k. We can see that this is the union of all the cylinder sets for the Φ^k such that $\Phi^k(l_{2k}) = 1$:

$$E_k = \bigcup_{\substack{\Phi^k \text{ s.t.} \\ \Phi^k(l_{2k})=1}} Cyl(\Phi^k) . \tag{3.3}$$

The time extent of event E_k is k.

A cylinder set is an event which corresponds to the history of the field up to a finite time. For each cylinder set, $Cyl(\Phi^n)$, the *class* operator, $C(\Phi^n)$ [9], for that finite history is given by

$$C(\Phi^n) \equiv P^H_{2n}(\Phi^n_{2n})\, P^H_{2n-1}(\Phi^n_{2n-1}) \ldots P^H_2(\Phi^n_2)\, P^H_1(\Phi^n_1) . \tag{3.4}$$

$P^H_a(\Phi^n_a)$ is the projection operator onto the eigenspace corresponding to the value, $\Phi^n_a = 0$ or 1, of Φ^n at link l_a, in the Heisenberg Picture:

$$P^H_a(\Phi^n_a) = U([(a+1)/2])^\dagger\, P_a(\Phi^n_a)\, U([(a+1)/2]) \tag{3.5}$$

where $P_a(\Phi^n_a)$ is the Schrödinger Picture projector, $U(k)$ is the unitary time evolution operator (3.2) and $[\cdot]$ denotes integer part. The Schrödinger picture projector is

$$P_a(\Phi^n_a) = |\Phi^n_a\rangle\langle\Phi^n_a| , \tag{3.6}$$

acting on the factor of H_q associated with l_a (tensored with the identity operator on the other factors).

Expressed in the Schrödinger Picture the class operator is

$$C(\Phi^n) = U(n) P_{2n}(\Phi^n_{2n})\, P_{2n-1}(\Phi^n_{2n-1})\, R_n \dots \\ \dots P_4(\Phi^n_4)\, P_3(\Phi^n_3)\, R_2\, P_2(\Phi^n_2)\, P_1(\Phi^n_1)\, R_1 \,, \tag{3.7}$$

which might be summarised by the slogan "evolve, project, evolve, project..."

We define a useful vector valued amplitude for the finite history Φ^n by applying its class operator to the initial state,

$$|\Phi^n\rangle \equiv C(\Phi^n)|\psi_0\rangle \,. \tag{3.8}$$

This vector is sometimes referred to in the literature as a "branch" [9].

The decoherence functional, D_q, is defined on cylinder sets by the standard expression [9]

$$D_q(Cyl(\Phi^n)\,; Cyl(\overline{\Phi}^m)) \equiv \langle \Phi^n |\, \overline{\Phi}^m\rangle \,. \tag{3.9}$$

The decoherence functional is defined on the whole event algebra, $\mathfrak{A}_q$, by additivity since all events are finite disjoint unions of cylinder sets. Although we have used the natural labelling that we chose for the vertices at the beginning, the decoherence functional thus constructed is independent of the chosen order and depends only on the vertices' causal order because the projectors and unitary evolution operators for spacelike separated vertices and links commute [10].

Note that the properties of the projectors ensure that the formula (3.9) for the decoherence functional is consistent with the condition of additivity when one cylinder set is a disjoint union of other cylinder sets. For example, $Cyl(\Phi^n)$ is a disjoint union of all events $Cyl(\Phi^{n+1})$ such that Φ^{n+1} agrees with Φ^n on the first $2n$ links and the decoherence functional of $Cyl(\Phi^n)$ (with any other event B) is indeed given as a sum:

$$D_q(Cyl(\Phi^n)\,; B) = \sum_{\substack{\Phi^{n+1} \text{ s.}t. \\ \Phi^{n+1}|_n = \Phi^n}} D_q(Cyl(\Phi^{n+1})\,; B)\,, \tag{3.10}$$

where the sum is over all four field configurations on the first $2(n+1)$ links which agree with Φ^n on the first $2n$ links.

If the initial state is a mixed state then the decoherence functional is a convex combination of pure state decoherence functionals.

This decoherence functional gives a level 2 measure, μ_q, on $\mathfrak{A}_q$ (see section 2).

3.2 The collapse model with the Bell ontology

The above unitary quantum field theory inspired a collapse model field theory [10] which, with the Bell ontology, can be understood as a level 1 (classical) measure theory in the Sorkin hierarchy (see section 2) as follows.

The space, Ω_c of all possible histories/formal trajectories is an identical copy of that for the quantum field theory, namely the set of all field configurations on the semi-infinite lattice to the future of σ_0. We will refer to field configurations in Ω_c as α in order to distinguish them from the elements of Ω_q which we refer to (as above) as Φ. The event algebra $\mathfrak{A}_c$ consists of finite unions of cylinder sets of elements of Ω_c and so is isomorphic to $\mathfrak{A}_q$.

The dynamics of the collapse model is given by a classical (level 1) measure. Since a level 1 measure is also level 2 – each level of the hierarchy includes the levels below it – a classical measure can also be given in terms of a decoherence functional and in this case the decoherence functional, D_c is given as follows.

Let α^n be a field configuration on the first $2n$ links. Define a vector valued amplitude $|\alpha^n\rangle \in H_q$ for each cylinder set $Cyl(\alpha^n)$:

$$|\alpha^n\rangle \equiv J_{2n}(\alpha^n_{2n})\; J_{2n-1}(\alpha^n_{2n-1})\; R_n \dots R_2\; J_2(\alpha^n_2)\; J_1(\alpha^n_1)\; R_1|\psi_0\rangle\,, \tag{3.11}$$

where $|\psi_0\rangle$ is the initial state on σ_0 and $J_a(\alpha^n_a)$ is the Kraus operator implementing a "partial collapse" onto the eigenspace corresponding to the value of α^n at link l_a. More precisely,

$$J_a(0) = \frac{1}{\sqrt{1+X^2}}(|0\rangle\langle 0| + X|1\rangle\langle 1|) \tag{3.12}$$

$$J_a(1) = \frac{1}{\sqrt{1+X^2}}\,(X|0\rangle\langle 0| + |1\rangle\langle 1|) \tag{3.13}$$

(where $0 \le X \le 1$) acting on the factor of H_q associated with link l_a (tensored with the identity operator for the other factors).

Then the decoherence functional, D_c is defined on cylinder sets by

$$D_c(Cyl(\alpha^n)\,; Cyl(\overline{\alpha}^n)) \equiv \langle \alpha^n|\,\overline{\alpha}^n\rangle \delta_{\alpha^n\,\overline{\alpha}^n}\,, \tag{3.14}$$

where $\delta_{\alpha^n\,\overline{\alpha}^n}$ is a Kronecker delta which is 1 if the two field configurations are identical on all $2n$ links and zero otherwise.

The decoherence functional is then extended to the whole event algebra, $\mathfrak{A}_q$ by additivity since all events are finite disjoint unions of cylinder sets. In particular, if $m > n$, the cylinder set $Cyl(\Phi^n)$ with time extent n is a disjoint union of cylinder sets with time extent m, and so it suffices to define D_q as above for cylinder sets of the same time extent: $D_c(Cyl(\alpha^n)\,; Cyl(\overline{\alpha}^m))$ is given by additivity.

Again, the decoherence functional thus constructed is independent of the chosen natural labelling and depends only on the vertices' causal order because of spacelike commutativity of the evolution operators and Kraus operators.

D_c is well-defined, in particular the additivity condition is consistent with the definition (3.14). For example, consider

$$D_c(Cyl(\alpha^n)\,;Cyl(\alpha^n))\,.$$

The event $Cyl(\alpha^n)$ is a disjoint union of all events $Cyl(\alpha^{n+1})$ for which α^{n+1} agrees with α^n on the first $2n$ links and indeed we have:

$$D_c(Cyl(\alpha^n)\,;Cyl(\alpha^n)) = \sum_{\substack{\alpha^{n+1} \text{ s.t.} \\ \alpha^{n+1}|_n=\alpha^n}} \sum_{\substack{\overline{\alpha}^{n+1} \text{ s.t.} \\ \overline{\alpha}^{n+1}|_n=\alpha^n}} D_c(Cyl(\alpha^{n+1})\,;Cyl(\overline{\alpha}^{n+1}))\,. \quad (3.15)$$

In verifying this, the crucial property is that of the Kraus operators: $J_0^2 + J_1^2 = 1$ and the fact that distinct histories have no interference, as expressed by the Kronecker delta. Note that without the Kronecker delta, equation (3.14) would not be a consistent definition of a decoherence functional satisfying additivity.

This decoherence functional is level 1 (classical): it satisfies

$$D_c(Y\,;Z) = D_c(Y\cap Z\,;Y\cap Z) \quad (3.16)$$

and this implies the Kolmogorov sum rule is satisfied by the measure μ_c defined by $\mu_c(Y) \equiv D_c(Y\,;Y)$. Being a level 1 measure, μ_c has a familiar interpretation as a probability measure. Indeed the measure μ_c defined on the cylinder sets is enough, via the standard methods of measure theory, to define a unique probability measure on the whole sigma algebra generated by the cylinder sets. There is, as yet, no analogous result for a quantal measure such as μ_q. Moreover, there is, as yet, no consensus on how to *interpret* a quantum measure theory. We will not address this important question here but refer to [6, 7, 13] for a new proposal for an interpretation of quantum mechanics within the framework of quantum measure theory.

3.3 Quantum and Classical

In every collapse model there is a coupling between classical stochastic variables and a quantum state. How is this classical-quantum coupling manifested in the generalised measure theory form of the lattice collapse model just given? We now show that there is indeed a quantum measure lurking within and we will expose the nature of the interaction of the quantal variables with the classical variables.

Consider a space of histories Ω_{qc} which is a direct product of the two spaces introduced above, $\Omega_{qc} = \Omega_q \times \Omega_c$, so that elements of Ω_{qc} are pairs of lattice field

configurations, (Φ,α). We will refer to the elements of Ω_q as *quantum histories/fields* and those of Ω_c as *classical histories/fields*. The event algebra $\mathfrak{A}_{qc}$ is the ring of sets generated by the cylinder sets, $Cyl(\Phi^n,\alpha^n)$, where the cylinder set contains all pairs (Φ,α) such that Φ coincides with Φ^n and α coincides with α^n on the first $2n$ links.

We now construct a decoherence functional on $\mathfrak{A}_{qc}$ by taking the unitary decoherence functional, D_q on $\mathfrak{A}_q$, defined above and "tying" the classical histories to the quantum histories by suppressing the decoherence functional by an amount that depends on how much the classical and quantum field configurations differ. The more they differ, the greater the suppression. In detail, define D_{qc} on $\mathfrak{A}_{qc}$ by first defining it on the cylinder sets:

$$\begin{aligned} D_{qc}(Cyl(\Phi^n,\alpha^n)\,;Cyl(\overline{\Phi}^n,\overline{\alpha}^n)) \equiv \\ D_q(Cyl(\Phi^n)\,;Cyl(\overline{\Phi}^n))\,\frac{X^{d(\Phi^n,\alpha^n)+d(\overline{\Phi}^n,\overline{\alpha}^n)}}{(1+X^2)^{2n}}\,\delta_{\alpha^n\,\overline{\alpha}^n} \end{aligned} \tag{3.17}$$

where $0\le X\le 1$ and $d(\Phi^n,\alpha^n)$ is equal to the number of links on which Φ^n and α^n differ. As usual it suffices to define D_{qc} for arguments which have the same time extent, n, because a cylinder set with time extent $m<n$ is a finite disjoint union of cylinder sets with time extent n. D_{qc} is extended to the full event algebra by additivity.

Checking that the definition (3.17) of D_{qc} on the cylinder sets is consistent with the property of additivity follows the same steps as for D_c and D_q. D_{qc} is level 2 in the Sorkin hierarchy, although it is clearly classical on Ω_c.

We now prove some lemmas regarding D_{qc} which lay bare the structure of our collapse model of a lattice field in histories form.

Lemma 1. *Let $(\Omega_q,\mathfrak{A}_q,D_q)$, $(\Omega_c,\mathfrak{A}_c,D_c)$ and $(\Omega_{qc},\mathfrak{A}_{qc},D_{qc})$ be defined as above for the lattice field theory. Then the decoherence functional for the collapse model, D_c is equal to D_{qc} coarse grained over Ω_q:*

$$D_c(A\,;\overline{A}) = D_{qc}(\Omega_q\times A\,;\Omega_q\times\overline{A})\quad\forall A,\overline{A}\in\mathfrak{A}_c\,. \tag{3.18}$$

Proof. It suffices to prove that

$$D_c(Cyl(\alpha^n)\,;Cyl(\overline{\alpha}^n)) = \sum_{\Phi^n,\overline{\Phi}^n} D_{qc}(Cyl(\Phi^n,\alpha^n)\,;Cyl(\overline{\Phi}^n,\overline{\alpha}^n))\,, \tag{3.19}$$

where the double sum is over all field configurations, Φ^n and $\overline{\Phi}^n$, on the first $2n$ links. The result follows by additivity because

$$\bigcup_{\Phi^n} Cyl(\Phi^n,\alpha^n) = \Omega_q\times Cyl(\alpha^n)\,. \tag{3.20}$$

Recall the definition of D_c,

$$D_c(Cyl(\alpha^n)\,; Cyl(\overline{\alpha}^n)) = \langle\alpha^n|\,\overline{\alpha}^n\rangle\delta_{\alpha^n\,\overline{\alpha}^n}\,,$$

where

$$|\alpha^n\rangle = J_{2n}(\alpha^n_{2n})\,J_{2n-1}(\alpha^n_{2n-1})\,R_n\ldots R_2\,J_2(\alpha^n_2)\,J_1(\alpha^n_1)\,R_1|\psi_0\rangle\,.$$

Each jump operator $J_a(\alpha^n_a)$ is a linear combination of the two projection operators $P_a(1) = |1\rangle\langle 1|$ and $P_a(0) = |0\rangle\langle 0|$ on link l_a (see equations 3.12 and 3.13). Substituting in this linear combination of projectors for each $J_a(\alpha^n_a)$ and expanding out, we see that the ket becomes a sum of 2^{2n} terms, one for each possible field configuration – call it Φ^n – on the $2n$ links. Each such term is precisely the vector valued amplitude $|\Phi^n\rangle$ (3.8) and each term is weighted by a factor

$$\frac{X^{d(\alpha^n,\Phi^n)}}{(1+X^2)^n}$$

from which the result follows. □

The next lemma shows that if we coarse grain D_{qc} over the classical histories instead, we find a quantum theory exhibiting the symptoms of environmental decoherence.

Lemma 2. *Define a decoherence functional $\widetilde{D}_q$ on Ω_q by*

$$\widetilde{D}_q(F\,;\overline{F}) \equiv D_{qc}(F\times\Omega_c\,;\overline{F}\times\Omega_c)\quad\forall F,\overline{F}\in\mathfrak{A}_q\,. \tag{3.21}$$

Then

$$\widetilde{D}_q(Cyl(\Phi^n)\,;Cyl(\overline{\Phi}^n)) = \left(\frac{2X}{1+X^2}\right)^{d(\Phi^n,\overline{\Phi}^n)} D_q(Cyl(\Phi^n)\,;Cyl(\overline{\Phi}^n))\,. \tag{3.22}$$

We leave the proof to the appendix. Note that the factor suppresses off-diagonal terms in the decoherence functional and so looks as if it is the result of environmental decoherence.

3.4 Equivalence to a model with environment

The system described by decoherence functional D_{qc} on the joint space Ω_{qc} was not derived from any physical consideration but simply invented as a way to unravel the decoherence functional of the collapse model. However, once obtained, the urge to coarse grain D_{qc} over the classical histories is irresistible and the "approximately diagonal" form of the resulting decoherence functional, $\widetilde{D}_q$ on Ω_q suggests it can be interpreted as having arisen from coupling to an environment.

Indeed, the mathematics of collapse models and of open quantum systems that result from coarse graining over an ignored environment are known to be closely related and so it is of no surprise to discover that our current model can be understood in this way. Indeed, the classical histories in the collapse model can simply be reinterpreted as histories of an environment consisting of variables, one per link, that interact impulsively with the field there, and then have no further dynamics.

Let the quantum lattice field, Φ, interact with a collection of environment variables, one for each link, taking values 0 or 1. The space of histories for the whole system is $\Omega_{qe} \equiv \Omega_q \times \Omega_e$, where the space of environment histories, Ω_e, is yet another copy of the same space of $\{0,1\}$-field configurations on the semi-infinite lattice. We denote an element of Ω_e by E, an environment configuration on the first $2n$ links by E^n, the corresponding cylinder set by $Cyl(E^n)$, and the value of the environment variable on link a by E^n_a.

In the standard state vector language, the Hilbert space of the whole system of field, Φ, and the environment is $H_{qe} \equiv H_q \otimes H_e$ where the environment Hilbert space, H_e, is an infinite tensor product of qubit Hilbert spaces, H_{e_a}, $a = 1, 2, \ldots$, one for each link l_a on the lattice to the future of σ_0.

Lemma 3. *There is a unitary dynamics of this system such that the unitary decoherence functional which encodes it, D_{qe}, is equal to D_{qc} if the environment histories are identified with the classical histories.*

Proof. The proof is by construction of such a dynamics. We add, to the unitary dynamics of the field Φ, a one-time interaction between Φ and the environment variable on each link which establishes a partial correlation between them. Since each environment state lives on exactly one link, it interacts only once and is then fixed, which means that the decoherence functional is diagonal on the environment histories.

We begin with the space of histories $\Omega_{qe} = \Omega_q \times \Omega_e$ and the Hilbert space $H_{qe} = H_q \otimes H_e$ where $H_e = \otimes_{a=1}^{\infty} H_{e_a}$ and each H_{e_a} is a qubit space.

The initial state is a tensor product:

$$|\Psi_0\rangle = |\psi_0\rangle_q \otimes_{a=1}^{\infty} |0\rangle_{e_a} \tag{3.23}$$

where $|\psi_0\rangle_q \in H_q$ is the same initial state for the field Φ as we had before.

After each elementary unitary evolution R_i is applied over vertex i, two unitary "partial measurement" operators U_{2i-1} and U_{2i} – to be defined – are applied to the Hilbert spaces associated with the outgoing links l_{2i-1} and l_{2i}, respectively.

Consider a single link, l_a. The factor of the total Hilbert space associated with l_a is the four-dimensional tensor product of the qubit space, H_{q_a}, of the Φ states on l_a and the qubit space H_{e_a}. In the field representation, the basis of this link Hilbert space is $\{|0\rangle_{q_a}|0\rangle_{e_a}, |1\rangle_{q_a}|0\rangle_{e_a}, |0\rangle_{q_a}|1\rangle_{e_a}, |1\rangle_{q_a}|1\rangle_{e_a}\}$.

The unitary partial measurement operator U_a is defined by its action on this basis:

$$\begin{aligned}
U_a|0\rangle_q|0\rangle_e &= \frac{1}{\sqrt{1+X^2}}\,|0\rangle_q\big(|0\rangle_e + X|1\rangle_e\big)\\
U_a|1\rangle_q|0\rangle_e &= \frac{1}{\sqrt{1+X^2}}\,|1\rangle_q\big(X|0\rangle_e + |1\rangle_e\big)\\
U_a|0\rangle_q|1\rangle_e &= \frac{1}{\sqrt{1+X^2}}\,|0\rangle_q\big(X|0\rangle_e - |1\rangle_e\big)\\
U_a|1\rangle_q|1\rangle_e &= \frac{1}{\sqrt{1+X^2}}\,|1\rangle_q\big(|0\rangle_e - X|1\rangle_e\big)\,,
\end{aligned} \tag{3.24}$$

where $0 \leq X \leq 1$ and we have suppressed the a label on all the kets. U_a acts as the identity on all other factors in the tensor product Hilbert space for the system.

The action of U_a is to leave Φ eigenstates alone and put the initial $|0\rangle_e$ environment state into a superposition of $|0\rangle_e$ and $|1\rangle_e$, so that the environment eigenstate that is correlated with the Φ eigenstate is relatively enhanced by a factor X^{-1}.

For each cylinder set $Cyl(\Phi^n, E^n)$ we define a vector valued amplitude, $|\Phi^n, E^n\rangle_{qe} \in H_{qe}$ by evolving the state over each vertex, applying the unitary partial measurements on the outgoing links and projecting onto the values of Φ^N and E^N on the links:

$$\begin{aligned}
|\,\Phi^n, E^n\rangle_{qe} \equiv Q_{2n}(E^n_{2n})\,P_{2n}(\Phi^n_{2n})\,Q_{2n-1}(E^n_{2n-1})\,P_{2n-1}(\Phi^n_{2n-1})&\\
U_{2n}\,U_{2n-1}\,R_n \ldots&\\
\ldots Q_2(E^n_2)\,P_2(\Phi^n_2)\,Q_1(E^n_1)\,P_1(\Phi^n_1)&\\
U_2\,U_1\,R_1|\Psi_0\rangle\,,&
\end{aligned} \tag{3.25}$$

where $|\Psi_0\rangle$ is defined in (3.23), $P_a(\Phi^n_a)$ is the projection operator onto the eigenspace corresponding to the value of Φ^n at link l_a and $Q_a(E^n_a)$ is the projection operator onto the eigenspace corresponding to the value of E^n at link l_a. $P_a(\Phi^n_a)$ is only non-trivial on the factor in H_q associated with link l_a and $Q_a(E^n_a)$ is only non-trivial on the factor in H_e associated with link l_a. As a consequence, the P projectors and Q projectors commute.

The initial state is a product, each U_a leaves Φ-eigenstates alone and the Q projectors act only on the environment states. We claim that therefore $|\Phi^n, E^n\rangle_{qe}$ is a product,

$$|\Phi^n, E^n\rangle_{qe} = |\Phi^n\rangle_q |E^n\rangle_e\,, \tag{3.26}$$

where $|\Phi^n\rangle_q \in H_q$ is the vector valued amplitude (3.8) for the plain vanilla unitary field theory and

$$|E^n\rangle_e = \frac{X^{d(\Phi^n,E^n)}}{(1+X^2)^n}\,|E_1^n\rangle_{e_1}|E_2^n\rangle_{e_2}\dots|E_{2n}^n\rangle_{e_{2n}} \tag{3.27}$$

where we have left off the factors of $|0\rangle$ for all the infinitely many links to the future of σ_n, which play no role.

The proof of this claim is given in the appendix.

The decoherence functional, D_{qe}, for the total system is given by

$$D_{qe}(Cyl(\Phi^n,E^n)\,;Cyl(\overline{\Phi}^n,\overline{E}^n)) \equiv \langle\Phi^n,E^n|\overline{\Phi}^n,\overline{E}^n\rangle \tag{3.28}$$

$$= \langle\Phi^n|\overline{\Phi}^n\rangle_q\langle E^n|\overline{E}^n\rangle_e\,. \tag{3.29}$$

Using (3.27), we see that the decoherence functional is zero unless $E^n = \overline{E}^n$ and we have

$$D_{qe}(Cyl(\Phi^n,E^n)\,;Cyl(\overline{\Phi}^n,\overline{E}^n)) = D_q(Cyl(\Phi^n)\,;Cyl(\overline{\Phi}^n))\,\frac{X^{d(\Phi^n,E^n)+d(\overline{\Phi}^n,\overline{E}^n)}}{(1+X^2)^{2n}}\,\delta_{E^n\,\overline{E}^n}\,. \tag{3.30}$$

As usual, we only need to define the decoherence functional for cylinder sets of equal time extent. We see that this is equal to D_{qc}, the decoherence functional of the collapse model (3.17). □

The model is technically unitary and so falls into the category of ordinary quantum theory, but the classicality of the environment variables is achieved by the device of postulating an infinite environment and one-time interactions.

4 Discussion

None of the physics we have presented is new. We have merely provided a novel perspective on a known model that arises when spacetime and histories are given a central role. Diósi stressed that both classical variables and quantum state are present in a collapse model and advocates ascribing reality to them both [14]. We have replaced the formalism of quantum state with quantum histories and by placing quantum and classical variables on the same footing in spacetime we can see more clearly the character of the interaction between them.

We claim that the structure outlined above for the collapse model for a lattice field theory, is generic to collapse models. There is always, more or less hidden in the model, a space of histories which is a product of a space of quantum histories and a space of classical histories, with a decoherence functional on it. For example,

in the case of the GRW model [15] the classical histories are countable subsets of Galilean spacetime, to the future of some initial surface, $t = 0$. The elements of such a countable subset are the "collapse centres" $(x_i, t_i), i = 1, 2, \ldots$. The probability distribution on these classical histories is given by a classical decoherence functional D_c, which is, essentially, set out in [16]. In order to follow the steps taken in this paper of unravelling D_c into D_{qc}, the positive operators, Gaussians, that correspond to the classical events are expressed as integrals of projection operators and the evolution between collapses expressed using the Dirac-Feynman propagator as a sum over the histories. The quantum histories, then, are precisely the histories summed over in the Dirac-Feynman path integral: all continuous real functions $\gamma : [0, \infty] \to \mathbb{R}$.

The continuum limit of the GRW model is the continuous spontaneous localisation model for a single particle [17, 18] and this too can be cast into the generic form as can be seen from the formulation of the model in terms of a "restricted propagator" as described in references [19, 20, 21]. Although the analysis in these references uses phase space path integrals, if it is the position operator whose eigenstates are collapsed onto, as is the case for the continuum limit of GRW, the path integrals can be transformed into configuration space path integrals. In this case, the quantum histories are again the continuous paths that contribute to the Dirac-Feynman sum-over-histories, but the classical histories are very noisy, and not continuous paths at all.

Note that in the lattice field theory the spaces of classical and quantum histories in this case are isomorphic, whereas in the GRW model and its continuum limit the quantum and classical histories are very different. In all cases, however, it is the quantum histories that bear all the consequence of dynamical law encoded in a local spacetime action, whereas the classical histories are simply dragged along by being tied to the quantum histories.

This state of affairs is illuminated further by considering coupling together two separate collapse models X and Y. Each model will contain both quantum and classical histories and the coupling between X and Y will be achieved by an appropriate term in the action involving the quantum histories alone. It is the quantum histories of X which directly touch the quantum histories of Y. The classical variables of X only react to the classical variables of Y because they are restricted to be close to the quantum variables which interact with the quantum variables of Y to which the classical variables of Y must, in their turn, be close.

The present authors believe, with Hartle, Sorkin and others, that a spacetime approach to quantum mechanics will be essential to progress in quantum gravity and

for this reason spacetime approaches should be carefully studied. Two important reasons for pursuing collapse models with the Bell ontology are that the models are already in spacetime form and the stochasticity involved is completely classical so all the familiar machinery of stochastic processes can be brought to bear: the stochasticity of collapse models causes no more interpretational difficulty than does the randomness of Brownian motion. The theory concerns the classical variables only and the quantum histories are relegated to some sort of auxiliary, hidden status, despite the fact that the dynamics of the model is most easily described in terms of these quantum histories. In order to pursue this direction, therefore, one must pay the price of ignoring the quantum histories as far as the ontology is concerned: "Pay no attention to that man behind the curtain" [22].

On the other hand, if the quantum histories are kept in the theory to be treated on the same footing, a priori, as the classical histories, then the question of the physical meaning of the quantum measure on them has to be wrestled with: what *is* the ontology in a quantum measure theory? But if *this* thorny problem is to be tackled, then one might start by trying to address it in the case of unitary quantum mechanics in the first instance. It may be that an interpretation of the quantum measure can be discovered that, by itself, provides a solution to the interpretational problems of quantum mechanics, while yet maintaining unitary dynamics and without need of new quantum-classical couplings.

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A Appendix

Proof. Of Lemma 2

Recall the definition of D_{qc}:

$$D_{qc}(Cyl(\Phi^n,\alpha^n)\,;Cyl(\overline{\Phi}^n,\overline{\alpha}^n)) = \\ D_q(Cyl(\Phi^n)\,;Cyl(\overline{\Phi}^n))\;\frac{X^{d(\Phi^n,\alpha^n)+d(\overline{\Phi}^n,\overline{\alpha}^n)}}{(1+X^2)^{2n}}\;\delta(\alpha^n,\overline{\alpha}^n)\;.$$

When the sum is taken over all α^n and $\overline{\alpha}^n$, field configurations on the first $2n$ vertices, it results in

$$D_{qc}(Cyl(\Phi^n) \times \Omega_c\,; Cyl(\overline{\Phi}^n) \times \Omega_c) = \frac{1}{(1+X^2)^{2n}}\; D_q(Cyl(\Phi^n)\,; Cyl(\overline{\Phi}^n)) \sum_{\alpha^n} X^{d(\Phi^n,\alpha^n)+d(\overline{\Phi}^n,\alpha^n)}\,. \tag{A.1}$$

Let $d(\Phi^n, \overline{\Phi}^n) = m$, which is the number of links on which the values of the two fields differ. For the duration of this proof only, we relabel the links on which the two fields differ $l_1, l_2, \ldots l_m$ and the rest, on which the fields agree, are labelled $l_{m+1}, \ldots l_{2n}$. Consider the exponent $d(\Phi^n, \alpha^n) + d(\overline{\Phi}^n, \alpha^n)$. The first m links contribute m to the exponent whatever α^n is, because for each link, α^n will agree with exactly one of Φ^n and $\overline{\Phi}^n$. Therefore

$$d(\Phi^n, \alpha^n) + d(\overline{\Phi}^n, \alpha^n) = m + 2\tilde{d}(\alpha^n, \Phi^n)\,, \tag{A.2}$$

where $\tilde{d}$ is the number of the last $2n - m$ links on which α^n and Φ^n differ.

The sum over α^n can be expressed as a multiple sum over the values of the α variable on each link in turn. We first do the sum over the values on the m links on which Φ^n and $\overline{\Phi}^n$ differ. The summand does not depend on the values on those links and so that gives a factor of 2^m

$$\sum_{\alpha^n} X^{d(\Phi^n,\alpha^n)+d(\overline{\Phi}^n,\alpha^n)} = 2^m X^m \sum_{\alpha^n_{m+1}} \cdots \sum_{\alpha^n_{2n}} X^{2\tilde{d}(\alpha^n,\Phi^n)}\,. \tag{A.3}$$

The remaining sum is over all α configurations on the last $2n-m$ links. There is one such configuration that agrees with Φ^n on all $2n - m$ links, $\binom{2n-m}{1}$ configurations that differ from Φ^n on one link, $\binom{2n-m}{2}$ that differ from Φ^n on two links, *etc.* The remaining sum therefore gives $(1 + X^2)^{2n-m}$ and we have

$$\sum_{\alpha^n} X^{d(\Phi^n,\alpha^n)+d(\overline{\Phi}^n,\alpha^n)} = 2^m X^m (1 + X^2)^{2n-m}\,, \tag{A.4}$$

and hence the result.

□

Claim 1.

$$|\,\Phi^n, E^n\rangle_{qe} = \frac{X^{d(\Phi^n,E^n)}}{(1+X^2)^n}\; |\Phi^n\rangle_q |E^n_1\rangle_{e_1} |E^n_2\rangle_{e_2} \ldots |E^n_{2n}\rangle_{e_{2n}} \otimes_{a=2n+1}^{\infty} |0\rangle_{e_a}\,, \tag{A.5}$$

where $|\Phi^n\rangle_q$ *is given by (3.1).*

This is the claim in lemma 3.

Proof. We use induction. It is trivially true for $n = 0$.

We assume it is true for n. Let $\Phi^{n+1}|_n = \Phi^n$ and $E^{n+1}|_n = E^n$. Then

$$| \Phi^{n+1}, E^{n+1}\rangle_{qe} = Q_{2n+2}(E^{n+1}_{2n+2})\, P_{2n+2}(\Phi^{n+1}_{2n+2})\, Q_{2n+1}(E^{n+1}_{2n+1})\, P_{2n+1}(\Phi^{n+1}_{2n+1}) \\ U_{2n+2}\, U_{2n+1}\, R_{n+1}|\Phi^n, E^n\rangle_{qe} \,. \tag{A.6}$$

The P projectors commute with the Q projectors. The P_a projectors also commute with the partial measurement operators U_a as can be seen from the definition of U (3.24). So we have

$$| \Phi^{n+1}, E^{n+1}\rangle_{qe} = \frac{X^{d(\Phi^n,E^n)}}{(1+X^2)^n}\, Q_{2n+2}(E^{n+1}_{2n+2})\, Q_{2n+1}(E^{n+1}_{2n+1})\, U_{2n+2}\, U_{2n+1} \\ \left[\, P_{2n+2}(\Phi^{n+1}_{2n+2})\, P_{2n+1}(\Phi^{n+1}_{2n+1})\, R_{n+1}|\Phi^n\rangle_q \,\right] \\ |E^n_1\rangle_{e_1} \ldots |E^n_{2n}\rangle_{e_{2n}} |0\rangle_{e_{2n+1}} |0\rangle_{e_{2n+2}} \otimes^{\infty}_{a=2n+3} |0\rangle_{e_a} \,. \tag{A.7}$$

The factor in square brackets is $|\Phi^{n+1}\rangle_q \in H_q$ and is unchanged by the U's because it is an eigenstate of the field Φ on the links l_{2n+1} and l_{2n+1}. The same factor is also unchanged by the Q's which only act on the environment states. U_{2n+1} turns $|0\rangle_{e_{2n+1}}$ into a linear combination of $|0\rangle_{e_{2n+1}}$ and $|1\rangle_{e_{2n+1}}$, enhancing the term which is correlated to the value Φ^{n+1}_{2n+1}. Similarly for U_{2n+2}. Finally $Q_{2n+1}(E^{n+1}_{2n+1})$ projects onto the state $|E^{n+1}_{2n+1}\rangle_{e_{2n+1}}$ and similarly for $Q_{2n+2}(E^{n+1}_{2n+2})$ with the result

$$| \Phi^{n+1}, E^{n+1}\rangle_{qe} = \frac{X^{d(\Phi^n,E^n)}}{(1+X^2)^n}\, \frac{X^{2-\delta(\Phi^{n+1}_{2n+2},E^{n+1}_{2n+2})-\delta(\Phi^{n+1}_{2n+1},E^{n+1}_{2n+1})}}{(1+X^2)}\, |\Phi^{n+1}\rangle_q |E^{n+1}\rangle_e \,. \tag{A.8}$$

The δ's in the exponent of X are Kronecker deltas and combining the factors of X gives the result.

□

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